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Letter to the Editors

Solution of Troesch's Two-Point Boundary Value Problem by Shooting Technique

1. INTRODUCTION

Recently Roberts and Shipman [1] have described a numerical investigation of Troesch's equation which describes confinement of a plasma column by radiation pressure. They have claimed that the Troesch's problem can be solved only by a combination of multipoint shooting with continuation and perturbation techniques and that none of these methods alone is capable of solving the problem. To overcome the difficulties associated with the problem, Jones [2] has proposed to modify the initial guess in such a way to avoid the overflow. The reason for the blow-up of the solution is the existence of a pole which for an unreasonable guess of the missing initial condition falls in the region of integration.

The present paper shows two techniques which render the numerical solution of the Troesch's equation almost trivial. The first method is based on an appropriate transformation which expands the length of the interval, the second method takes use of the exchange of independent for dependent variables. The former approach is sometimes referred to as the parameter mapping technique.

2. PARAMETER MAPPING TECHNIQUE

The Troesch's problem is in the form:

$$d^2y/dt^2 = n \sinh ny \tag{1a}$$

subject to the boundary conditions

$$y(0) = 0, \qquad y(1) = 1. \tag{1b}$$

The aim of the solution of (1a) and (1b) is to find the profiles and the missing initial conditions for different values of the parameter n.

After substitution,

$$ny = w, \qquad nt = x, \tag{2}$$

the parameter n can be eliminated from (1a),

$$d^2w/dx^2 = \sinh w; \tag{3}$$

however, it appears in the boundary conditions

$$w(0) = 0, \qquad w(n) = n. \tag{4}$$

The mapping technique is very simple; on choosing the missing initial condition

$$dw(0)/dx = \eta \; (= dy(0)/dt) \tag{5}$$

(3) is integrated using the initial conditions (5) and $w(0) = 0$ until

$$w(n) = n. \tag{6}$$

One has to iterate to satisfy (6) within a preassigned tolerance, ϵ, e.g. if $w(n) > n$ the integration procedure returns to the preceding value and the integration step is halved. This simple bisection procedure is repeated as long as the difference $w(n) - n > \epsilon$. However a straightforward method of satisfying (6) exists. Starting from a certain value x_0, where $dw(x_0)/dx > 1$, one can switch from the integration of

$$\begin{aligned} dw/dx &= z, \qquad dz/dx = \sinh w, \\ w(0) &= 0, \qquad z(0) = \eta, \end{aligned} \tag{7}$$

which yields $z(x_0) = z_0$ and $w(x_0) = w_0$, to a new system

$$dx/d\varphi = 1/(z-1), \qquad dw/d\varphi = z/(z-1), \qquad dz/d\varphi = (\sinh w)/(z-1), \tag{8}$$

subject to the initial conditions

$$\varphi = w_0 - x_0: \qquad x = x_0, \quad w = w_0, \quad z = z_0. \tag{9}$$

Clearly, the value of w for $\varphi = 0$ is the unknown value of the parameter n.

The dependence calculated for a sequence of η is presented in Table I. The integration was performed in the double precision arithmetics on the Tesla 200 computer, i.e., with the precision of 15 significant digits, using the Runge–Kutta–Merson marching integration technique with the automatic step-size control [3]. A numerical experiment pointed out that the direct algorithm is more effective than the bisection method. For illustration purposes, a course of one integration is presented in Table II. It is obvious that this algorithm for a given value of η calculates a posteriori the value of n. It is very simple to interpolate in Table I if

TABLE I

Dependence of $y'(0)$ on n Obtained by the Parameter Mapping Method.
(For given $y'(0)$ the value of n is calculated)

$\eta = y'(0)$	n
0.9	0.792
0.8	1.151
0.6	1.753
0.4	2.394
0.2	3.308
0.1	4.129
0.0457	5.000
0.01	6.611
0.003	7.849
0.001	8.965
0.000356	10.01
0.0001	11.28
10^{-5}	13.59
10^{-6}	15.89
10^{-7}	18.20
10^{-8}	20.50
10^{-9}	22.80
10^{-10}	25.10
10^{-11}	27.41
10^{-12}	29.71

one needs to know the value $y'(0)$ for a given value of n. Looking at Table I, an approximate dependence can be easily evaluated:

$$n \doteq 29.71 - \ln 10^{12} y'(0), \tag{10}$$

which is valid roughly for $y'(0) < 10^{-4}$. This asymptotic relation corresponds approximately to the relation

$$y'(0) \doteq 8e^{-n}. \tag{11}$$

It is shown in [1] that the associated initial-value problem for this particular

TABLE II

A Course of One Integration of Eqs. (7) and (8) in Parameter Mapping Technique

x	w	$z = dw/dx$	
0.0000	0.0000	0.0010	
3.5933	0.0182	0.0182	
6.0402	0.2102	0.2106	
7.0135	0.5594	0.5667	
7.7081	1.1433	1.2065	
8.1071	1.7653	2.0036	
8.5071	2.8918	4.0101	
8.7886	4.6259	10.0053	
φ	x	w	z
−4.1627	8.7886	4.6259	10.005
−3.6010	8.8414	5.2404	13.666
−2.8809	8.8879	6.0070	20.107
−2.0032	8.9244	6.9212	31.804
−1.0061	8.9495	7.9435	53.057
0.0000	8.9646	8.9646	88.426

value of $y'(0)$ possesses a pole approximately at $t = 1$. Usually for a guess $y'(0) > 8e^{-n}$ the pole lies in the integration range. Hence for higher values of n the unmodified shooting technique cannot be easily used.

On the other hand, for low values of n, an approximation can be developed, too. The implicit solution possessed by Troesch's problem [1] is

$$t = \int_0^y \frac{dv}{(2\cosh nv + C)^{1/2}}\,. \tag{12}$$

To evaluate the integration constant C, we have

$$\int_0^1 \frac{dv}{(2\cosh nv + C)^{1/2}} = 1. \tag{13}$$

Differentiation of (12) yields

$$y'(0) = (2 + C)^{1/2}, \tag{14}$$

because $y(0) = 0$. For low values of n the term $\cosh nv$ can be approximated by the truncated Taylor series:

$$\cosh nv \doteq 1 + \tfrac{1}{2}n^2v^2.$$

After substitution of this approximation into (13) and using the relation

$$\frac{1}{n}\int_0^1 \frac{dv}{\left(\frac{2+C}{n^2} + v^2\right)^{1/2}} = \left[\frac{1}{n}\ln\left| v + \left(\frac{2+C}{n^2} + v^2\right)^{1/2}\right|\right]_0^1,$$

we have

$$1 + \left(1 + \frac{2+C}{n^2}\right)^{1/2} = \left(\frac{2+C}{n^2}\right)^{1/2} e^n.$$

After combination with (14) we obtain the relation ($n \geqslant 1$)

$$y'(0) \doteq 2ne^{-n}/(1 + e^{-2n}) \doteq 2ne^{-n}, \tag{15}$$

which is more accurate for $n < 4$ than (10) or (11) (for $n = 2.349$ the resulting $y'(0)$ is 0.73 or 0.43 resulting from (11) or (15), respectively; cf. Table I.).

3. Change of Variables

The appropriate strategy of solving the nonlinear problems containing a pole is to perform a transformation which makes it possible to eliminate it. Obviously, the exchange of dependent for independent variables allows us to eliminate the pole provided the dependent variable is monotonic.

Let us rewrite (1) to a set of first-order differential equations:

$$dy/dt = z, \qquad dz/dt = n \sinh ny. \tag{16}$$

For a monotonic function $y(t)$, i.e., for $z \neq 0$, the inversion of variables yields:

$$dt/dy = 1/z, \qquad dz/dy = (n \sinh ny)/z, \tag{17}$$

subject to initial conditions

$$y = 0: \qquad t = 0, \quad z = \eta. \tag{18}$$

The missing value η must be guessed in such a way that after marching integration across, (19) is satisfied within a predetermined tolerance:

$$y = 1: f(\eta) = t - 1 = 0. \tag{19}$$

TABLE III

Results for Method of Section 3.
(Two Different Initial Guesses for the Iteration Process are Presented.)[a]

n	k	$\eta_k(=y'(0))$	λ	$\eta_k(=y'(0))$	λ
	0	0.10000		0.01000	
	1	0.02168	1	0.02520	1
5	2	0.03786	1	0.04023	1
	3	0.04503	1	0.04540	1
	4	0.04574	1	0.04575	1
	5	0.04574	1	0.04575	1
	0	0.100*E*-2		0.100*E*-3	
	1	0.487*E*-3	1/2	0.228*E*-3	1
10	2	0.338*E*-3	1	0.331*E*-3	1
	3	0.358*E*-3	1	0.357*E*-3	1
	4	0.358*E*-3	1	0.358*E*-3	1
	0	0.100*E*-3		0.100*E*-4	
	1	0.722*E*-5	1/4	0.295*E*-5	1/2
	2	0.331*E*-5	1/2	0.239*E*-5	1
15	3	0.231*E*-5	1	0.244*E*-5	1
	4	0.244*E*-5	1	0.244*E*-5	1
	5	0.244*E*-5	1		
	0	0.100*E*-5		0.100*E*-6	
	1	0.487*E*-6	1/8	0.987*E*-8	1/2
20	2	0.748*E*-7	1/4	0.149*E*-7	1
	3	0.182*E*-7	1/2	0.164*E*-7	1
	4	0.164*E*-7	1	0.165*E*-7	1
	5	0.165*E*-7	1		

[a] The Newton method is used in the form $\eta_{k+1} = \eta_k - \lambda f(\eta_k)/f'(\eta_k)$ for solution of Eq. (19). The initial choice is $\lambda = 1$. If the condition $|f(\eta_{k+1})| < |f(\eta_k)|$ is not fulfilled the step λ is halved.

To solve the nonlinear boundary value problem (17)–(19) a modification of the shooting technique may be adopted. In our calculations we have used the shooting method with the Newton root-finding algorithm. To calculate the values of derivatives for the Newton method a set of auxiliary equations must be solved simultaneously with the original differential equation [3]. To integrate the relevant initial-value problems Merson's modification of the Runge–Kutta method has been used. Some results of computation with this technique are reported in Table III. It is obvious that after this transformation the integration of the Troesch's problem is very simple.

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